CONSISTENCY AND SATISFIABILITY

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Abstract. In this talk I will discuss the theoretical connection on results in my recent works [2],[3],[4]. The study of Classical Model Existence Theorem (every consistent set is classically satisfiable) offers not only I the strange situation that a lot of weak logics have the same consistency as the classical logic has, but also that it provides a theoretical ground for the study of the general theory of consistency. An easy consequence from this study, which is a sharpened Glivenko’s theorem, is also presented.

What can we achieve from the study of classical model existence property?

In classical logic the extended completeness theorem (for any $\Sigma$ and $\varphi$, $\Sigma \models \varphi$ implies $\Sigma \vdash \varphi$) is obviously an extension of completeness theorem (which is just the special case $\Sigma = \emptyset$). And for any classically sound logic $L$, if $L$ has all classical tautologies as its theorems and it has the rule Modus Ponens $MP$, then such $L$ is exactly (a proof system of) the classical logic. This is achieved by compactness theorem (a result based on classical semantics), the semantic version of deduction theorem, and $MP$: $\Sigma \models \varphi \implies (by$ compactness) there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0 \models \varphi \implies (by$ semantic deduction theorem) $\models \bigwedge \Sigma_0 \rightarrow \varphi \implies (by$ completeness) $\vdash \bigwedge \Sigma_0 \rightarrow \varphi \implies (by$ $MP) \Sigma_0 \vdash \varphi \implies (and$ then by monotonicity) $\Sigma \vdash \varphi$.

Since to prove the compactness theorem for classical semantics one may need some extra effort (e.g. ultraproduct), it seems more convenient to prove

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the extended completeness theorem directly. (In this way the compactness theorem is easily proved since classical logic as a proof system is finitary: \( \Sigma \vdash \varphi \) implies there is a finite subset \( \Sigma_0 \subseteq \Sigma \) such that \( \Sigma_0 \vdash \varphi \).)

The direct proof of extended completeness theorem for classical logic, as in most logic textbooks, is done by proving the following two statements:

\((CME)\) Every consistent\(^1\) set has a classical model (under the standard two-valued truth-functional semantics).

\((RAA)\) If \( \Sigma \not\vdash \varphi \), then \( \Sigma \cup \{ \neg \varphi \} \) is consistent.

Let us call the first statement the classical model existence property \((CME\) for short) and the second statement the Reducio Ad Absurdum property \((RAA\) for short). Though \( CME \) holds in classical logic, classical logic is not the only logic satisfying \( CME \) (see [2]). The reason is that weak logics may have \( CME \) true but \( RAA \) false. What may be interesting is that such logics cannot be distinguished from classical logic by any example of consistent sets, i.e., in any of these logics (including classical logic) for any set \( X \) of sentences, whether \( X \) can derive a contradiction does not depend on which of these logics\(^2\) we choose.

However, even one does not appreciate the fact that there are logics which have the same consistency as classical logic has, the author would like to offer another reason why should we study \( CME \): the study of \( CME \) is related to the study of consistencies.

Then what is consistency? We take the following approach. Consider a given semantics \( M \) and let \( U \) be the collection of all unsatisfiable sets with respect to \( M \). To a logic \( L \) a set \( X \) is inconsistent iff from \( X \) we derive something wrong (with respect to \( L \)). However, there are many choices of something wrong: it can be the set of all sentences (usually we call it the

\(^1\)Here we can have a lot of choices about consistencies, which means (in a fixed proof system) the unprovability of certain types of sets from the given premise(s). We will discuss about this later.

\(^2\)Here logics mean proof systems.
absolute consistency), the set \{\bot\} where \bot is the sentence which is always false (here the author calls it the \bot-consistency), any set of the form \{p, \neg p\} for any sentence \(p\) (usually we call it the simple consistency), etc. We then have the following definition (the reader may consider just the classical semantics as an example).

**Definition 1.**  
(1) Let \(L\) be a proof system, \(S\) be the set of all sentences. For any \(X, Y \subseteq S\), we say \(X\) derives \(Y\) by \(L\), in symbol \(X \vdash_L Y\), iff \(X \vdash_L \varphi\) for every \(\varphi \in Y\).

(2) Let \(M\) be a semantics, \(U\) be the collection of all unsatisfiable sets with respect to \(M\), the non-empty set \(\Gamma \subseteq U\). We say that a set \(X\) is \(\Gamma\)-consistent with respect to \(L\) iff for any \(Y \in \Gamma\), it is not the case that \(X \vdash_L Y\).

According to this definition, the weakest consistency is the \(U\)-consistency: Let \(\Gamma \subseteq U\), if (to any \(L\)) the set \(X\) is \(\Gamma\)-inconsistent, then \(X\) is clearly \(U\)-inconsistent (since \(\Gamma \subseteq U\)). On the other hand, since \(S\) (the set of all sentences) is the \(\subseteq\)-greatest element in \(U\), to any \(L\) and any \(\emptyset \neq \Gamma \subseteq U\) if the set \(X\) is \(\{S\}\)-inconsistent, then \(X\) is \(\Gamma\)-inconsistent (since \(X \vdash_L S\) implies \(X \vdash_L Y\) for any \(Y \in \Gamma\)). That is, \(\{S\}\)-consistency (the absolute consistency) is the strongest consistency (to any \(L\)).

With this definition, we reconsider \(CME\) with respect to certain consistency. First let \(CU\) be the collection of all classically unsatisfiable sets and a nonempty \(\Gamma \subseteq CU\). We then may restate \(CME\) as a property about two consistencies (\(\Gamma\)-consistency and \(CU\)-consistency):

\((CME\ \text{w.r.t.} \ \Gamma\text{-consistency})\) If \(X\) is \(\Gamma\)-consistent, then \(X\) is \(CU\)-consistent (i.e., \(X\) is not classically unsatisfiable).
Since classical unsatisfiability is the weakest consistency\(^3\), when \(L\) has \(CME\) with respect to \(\Gamma\)-consistency, it means that \(\Gamma\)-consistency is equivalent to \(CU\)-consistency in \(L\).

In classical logic it is the case that all consistencies are equivalent: we can take \(\bot\)-consistency, then \(CME\) with respect to \(\bot\)-consistency implies that \(CU\)-consistency is equivalent to \(\bot\)-consistency in classical logic. On the other hand, since \(\bot \rightarrow A\) is a classical tautology, \(\bot\)-consistency is equivalent to \(\{S\}\)-consistency in classical logic. Then it is clear that in classical logic all consistencies coincide.

However, to have a logic \(L\) satisfying \(CME\) with respect to some \(\Gamma\)-consistency it does not matter that whether \(L\) has all of its consistencies (with respect to classical semantics) equivalent. Let me explain what the result in [2] means.

**Theorem.** Theorem 17, [2] and a (well-known) result in Section 3, [2]

1. \(CME\) with respect to \(\bot\)-consistency holds in the \(\{\rightarrow, \bot\}\)-fragment of intuitionistic logic, \(\mathcal{H}(DT1, DT2, ECQ; MP)\). (In this case, all consistencies coincide.)

2. \(CME\) with respect to \(\bot\)-consistency holds in the paraconsistent logic \(\mathcal{H}(DT1, DT2, DA^*_\bot; MP)\). (In this case, \(\bot\)-consistency is equivalent to \(CU\)-consistency, but not equivalent to absolute consistency.)

Here the axiom schemes are

\[
\begin{align*}
(DT1) & \quad A \rightarrow (B \rightarrow A) \\
(DT2) & \quad [A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)] \\
(ECQ) & \quad \bot \rightarrow A \\
(DA^*_\bot) & \quad (A \rightarrow \bot) \rightarrow \{(A \rightarrow B) \rightarrow \bot \} \\
(DN_{\bot}) & \quad [(A \rightarrow \bot) \rightarrow \bot] \rightarrow A
\end{align*}
\]

\(^3\)This is the case that we take classical semantics. In this case, \(U = CU\) is the weakest consistency.
Therefore, every weak logic $L$ with $CME$ is not necessary to have all consistencies equivalent. We may have a paraconsistent logic $\mathcal{H}(DT_1, DT_2, DA^*_\bot; MP)$ and the $\{\rightarrow, \bot\}$-fragment of classical logic have the same $\bot$-consistency, though $CME$ w.r.t. absolute consistency does not hold in $\mathcal{H}(DT_1, DT_2, DA^*_\bot; MP)$. It seems strange that (assuming that $\mathcal{H}(DT_1, DT_2, DA^*_\bot; MP)$ has a Kripke semantics so that it is sound and complete) when we look at the Kripke model of $\{p, p \rightarrow \bot, \bot\}$ (at one node $w$ of that Kripke model), it is obvious that $w$ is not a classical model. However, $CME$ with respect to $\bot$-consistency tells us that if we only look at $X$ which can not derive $\bot$ in $\mathcal{H}(DT_1, DT_2, DA^*_\bot; MP)$, $X$ definitely has a classical model, i.e., $CME$ works at the part of $\bot$-unprovable sets in $\mathcal{H}(DT_1, DT_2, DA^*_\bot; MP)$.

Then does this offer us any good result? Of course. We simply have the following Glivenko’s theorem.

**Theorem 2.** If $\Sigma \vdash \varphi$ in $\mathcal{H}(DT_1, DT_2, DN_\bot; MP)$, the $\{\rightarrow, \bot\}$-fragment of classical propositional logic, then $\Sigma \vdash (\varphi \rightarrow \bot) \rightarrow \bot$ in $\mathcal{H}(DT_1, DT_2, DA^*_\bot; MP)$.

**Proof.** If not, by deduction theorem $\Sigma \cup \{\varphi \rightarrow \bot\}$ is $\bot$-consistent. By $CME$, $\Sigma \not\models \varphi$, a contradiction. ☐

To the possible bias that the study of $CME$ is just a trivial consequence of Glivenko’s theorem for intuitionistic propositional logic (and Glivenko’s theorem for intuitionistic propositional logic is proved by syntactic translation), above theorem shows that it is not so. And our proof has no syntactic translation. Therefore, whether there is a syntactic translation for above theorem (and what kind of syntactic translation if it exists) is one of the topics raised from the study of $CME$.

**Concluding Remark:** Not only $\rightarrow$ can be weakened as above, but also $\land, \lor$ can be weakened. We can easily enlarge above result (see [2]) to $L(\rightarrow, \bot, \land, \lor)$ and the way of weakening axioms actually suggests 8 possible weakened $\land$’s and 16 possible $\lor$’s. It can be proved that to some $BCI$ or $BCIW$ extension $L$, $L$ satisfies $CME$. And it is applicable to not only
in propositional case, but also in predicate case (see [3]). There are also corresponding Glivenko like theorem. However, the author sees no need to present any of these here (if above result is not understood nor appreciated). (And if you wish to know these results, you may need to wait for my writings.)

REFERENCES


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