Abstract: In this talk we investigate the compactness theorem (as a property) in non-classical logics. We focus on the following problems: (a) What kind of semantics make a logic having compactness theorem? (b) What is the relationship between the compactness theorem and the classical model existence theorem (CME)/model existence theorem?

1. Introduction

Compactness theorem, which states that every finite subset of $\Sigma$ has a model if and only if $\Sigma$ has a model, is quite useful in first order logic (for example, with it and the strong completeness theorem\textsuperscript{1} one can easily prove

\textsuperscript{1}The strong completeness theorem states that every consistent set has a classical model.

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the existence of nonstandard arithmetic models and the upward Löwenheim-
Skolem Theorem, etc.). However, its proofs in most logic textbooks are lim-
ited to CPL (classical propositional logic) or FOL (first order logic). This
somehow limits our vision to see how compactness theorem, as a metalogical
property, can be proved in other logics and delays the investigation on the
relationship between semantics of these logics.

In most logic textbooks or model theory books the compactness theorem
are usually proved in some of the following three ways:

(1) Compactness is proved by maximal extension (in the CPL case) or
   by adding new constants (Henkin’s method) to construct Hintikka
   sets (in the FOL case).

(2) Compactness is proved as an easy corollary of the extended com-
   pleteness theorem (actually the classical model existence theorem
   will suffice).

(3) Compactness is proved by the ultraproduct construction.

In this talk I will focus on the propositional cases of the first two ap-
proaches and show that these two approaches offer us some sufficient condi-
tions to prove compactness theorem for other logics in an abstract way.

Remark 1. To make the readers have some appreciation on what is done in
this paper, I suggest that the readers may try to prove the compactness the-
orems for finite-valued propositional logics or the intuitionistic propositional
logic (IPL).
2. Expanding to a maximal set versus expanding to a total truth assignment

Recall that a set $\Sigma$ is finitely satisfiable iff every finite subset of $\Sigma$ has a model. In [4], the compactness theorem is proved as follows:

Suppose that $\Sigma$ is finitely satisfiable. Enumerate all sentences in that language, say, $\varphi_1, \ldots, \varphi_n, \ldots$. Define $\Delta_0 = \Sigma$ and

$$\Delta_{n+1} = \begin{cases} 
\Delta_n \cup \{\varphi_n\} & \text{if } \Delta_n \cup \{\varphi_n\} \text{ is finitely satisfiable}; \\
\Delta_n \cup \{\neg\varphi_n\} & \text{else}.
\end{cases}$$

And we need a technical lemma:

**Lemma 2.** For any sentence $\alpha$, if a set $\Pi$ is finitely satisfiable, then either $\Pi \cup \{\alpha\}$ is finitely satisfiable or $\Pi \cup \{\neg\alpha\}$ is finitely satisfiable.

With the aid of Lemma 2, one can easily check the following facts:

1. $\Delta_n$ is finitely satisfiable for all $n$.
2. $\Sigma \subseteq \bigcup_n \Delta_n$.
3. For all $\alpha$, either $\alpha \in \Delta = \bigcup_n \Delta_n$ or $\neg\alpha \in \Delta$.
4. $\Delta$ is finitely satisfiable.
5. The interpretation $\mu$, which has $\mu(p) = T$ iff $p \in \Delta$ for any atomic sentence $p$, is a model of $\Delta$.

**Remark 3.** However, it is not clear how above proof can be modified to prove the compactness theorem for any finite-valued truth-functional propositional
logic. Note that above proof assumes that \( \neg \) is in the propositional language, but this is not really necessary\(^2\). Since what we need is exactly a truth assignment (i.e., an interpretation), it is better to have extensions on (partial) truth assignments rather than sets of sentences.

Now we define some basic notions. Let \( S = \{ p_0, \ldots, p_n, \ldots \} \) be the set of all atomic sentences. A partial truth assignment \( f \) is a relation \( f \subseteq S \times \{ T, F \} \) which is also a function with \( \text{Dom}(f) \subseteq S \). Now we extend the notion of finite satisfiability to what follows.

**Definition 4.**  
(1) A set \( \Pi \) is satisfiable with respect to a partial truth assignment \( f \) iff there is a total truth assignment \( g \) such that \( f \subseteq g \) and \( g \) is a model of \( \Pi \).

(2) \( \Pi \) is finitely satisfiable with respect to a partial truth assignment \( f \) iff every finite subset of \( \Pi \) is satisfiable with respect to \( f \).

Note that “a set \( \Pi \) is finitely satisfiable” is now read as “\( \Pi \) is finitely satisfiable with respect to the empty set \( \emptyset \) (as a partial truth assignment).”

With this definition, to a \((n+1)\)-valued semantics we can prove the following lemma, which is a generalization of Lemma 2.

**Lemma 5.** To any \((n+1)\)-valued semantics with \( V = \{ 0, \ldots, n \} \) (the set of all its truth values) and \( D \subseteq V \) (the set of designated truth values), if

\(^2\)We will see that in finite-valued semantics we do not really need to use anything like a negation.
Π is finitely satisfiable with respect to \( f \), and atomic sentence \( p_i \notin \text{Dom}(f) \), then there is some \( j \in V \) such that \( \Pi \) is finitely satisfiable with respect to the extension \( f' = f \cup \{ (p_i, j) \} \). (Note that in finite-valued semantics the set \( \Pi \) is satisfiable iff there is a total truth assignment \( f : S \to \{0, \ldots, n\} \) such that the truth value of any \( \varphi \) in \( \Pi \) by applying \( f \) is in \( D \).)

**Proof.** (Sketch:) The key idea is that if for every \( j \in V \) there is a finite subset \( \Sigma_j \subseteq \Sigma \) such that \( \Sigma_j \) is not satisfiable with respect to \( f \cup \{ (p_i, j) \} \), then the finite union \( \bigcup_{j=1}^{n} \Sigma_j \), as a finite subset of \( \Sigma \), is not satisfiable with respect to \( f \), a contradiction. \( \square \)

Now we can prove the compactness theorem for any finite-valued truth-functional semantics (with the setting in Lemma 5). Let \( \Sigma \) be finitely satisfiable in such a semantics. Instead of enlarging the set \( \Sigma \) to a maximal extension \( \Delta \), we construct a sequence of partial truth assignments. Define that \( f_0 = \emptyset \) and \( f_{n+1} = f_n \cup \{ (p_n, j) \} \), where \( j \) is the smallest value in \( V \) such that \( \Sigma \) is finitely satisfiable with respect to \( f_{n+1} \). The existence of \( j \) is guaranteed by Lemma 5. Then \( \bigcup_{n=0}^{\infty} f_n \) is a total truth assignment and a model of \( \Sigma \).

**Remark 6.** This is inspired by the proof in [1] (pages 45-6), which leads us to the construction of a truth assignment (as in Lemma 5) instead of the construction in [4] (pages 59-60). Surely to any finite-valued truth-functional semantics one can firstly expand the set of logical connectives (so that the intended model or interpretation can be expressed by a maximal
set) and then prove the compactness theorem by expanding $\Sigma$ to a maximal extension. However, in propositional cases the way of constructing truth assignment is easier because there is no need to assume that the intended model or interpretation can be expressed by the given logical connectives (e.g., in $\text{CPL}$ $\neg$ is used to expressed the choice of truth values). In the predicate case, the expansion of the formal language is useful (though one can still say not necessary) for the model constrction.

**Remark 7.** Lemma 5 can be seen as a sufficient condition to prove compactness theorem for other logics. However, whether one can use this method to prove the compactness theorem for $\text{IPL}$ (intuitionistic propositional logic) is not clear at all. Note that Gödel proved that $\text{IPL}$ is not realized by any finite-valued semantics (see [6], pages 222-5).

3. **ON PROVING COMPACTNESS BY COMPLETENESS**

In [13], the way of proving compactness by completeness theorem is criticized as “an error of method.” Since the compactness theorem is purely a model-theoretic statement, methodologically there is no reason why one has to use any proof system to prove compactness.

Anyway, what we try to do here is to see to what logic we can apply this “proving-compactness-by-completeness” method.

For simplicity we pick a sentence constant $\bot$ which has a fixed truth value from $V \setminus D$ (here $V, D$ may be infinite sets). Then the model existence theorem, which states that every $\bot$-consistent set has a model, can be stated
as “For any \( \Sigma \), \( \Sigma \models \perp \) implies \( \Sigma \vdash \perp \). (Here \( \models \) means the validity with respect to the corresponding semantics.)

Now consider a proof system \( \vdash \) with a semantics \( \models \) such that \( \vdash \) is sound to \( \models \) and the model existence theorem holds. If \( \Sigma \) is finitely satisfiable (with respect ot \( \models \)), then either \( \Sigma \not\vdash \perp \) or \( \Sigma \vdash \perp \).

In the first case, \( \Sigma \not\models \perp \) by the model existence theorem. Then there is a model \( \mu \) which satisfies \( \Sigma \) but does not satisfy \( \perp \). Hence \( \Sigma \) is satisfiable.

In the second case \( \Sigma \vdash \perp \), if \( \vdash \) is finitary and sound to \( \models \), we can then show that there is a finite subset \( \Sigma_0 \subseteq \Sigma \) such that \( \Sigma_0 \vdash \perp \). By soundness \( \Sigma_0 \models \perp \). Then \( \Sigma_0 \) is not satisfiable and \( \Sigma \) is not finitely satisfiable, a contradiction.

We summarize this as follows.

\textbf{Theorem 8}. If a proof system \( \vdash \) is sound to a semantics \( \models \), \( \perp \) is in the language of the system \( \vdash \), \( \vdash \) is finitary (that is, \( \Sigma \vdash \varphi \) implies there is a finite subset \( \Sigma_0 \subseteq \Sigma \) such that \( \Sigma_0 \vdash \varphi \)) and the model existence theorem (for any \( \Sigma \), \( \Sigma \models \perp \) implies \( \Sigma \vdash \perp \)) holds, then the compactness theorem for \( \models \) holds.

Therefore the compactness theorem for a lot of semantics (as listed in [5]) hold by the easy argument in Theorem 8. However, one may think that now we appeal to a more difficult theorem Model Existence Theorem or another even more difficult Extended Completeness theorem (\( \Sigma \models \varphi \) implies \( \Sigma \vdash \varphi \))
for any $\Sigma, \varphi$). And to look for a semantics for a proof system $\vdash$ so that soundness and completeness hold seems difficult.

Actually it is not difficult\textsuperscript{3} to give a semantics for arbitrary proof system. We can define $V$ to be the set of all sentences, and the truth value of a sentence $\varphi$ is $\varphi$ itself. The truth functions of connectives can be understood by the following example: The value of $(\varphi \lor \psi)$, again $(\varphi \lor \psi)$, is computed by the truth function $f(x, y) = (x \lor y)$. (One can check, by mathematical induction, that $\mu((\varphi \lor \psi)) = f(\mu(\varphi), \mu(\psi)) = f(\varphi, \psi) = (\varphi \lor \psi)$.) Next we associate the truth assignment with $\vdash$. For any set $\Sigma$, we define $\bar{\Sigma} = \{ \varphi \mid \Sigma \vdash \varphi \}$ as a truth assignment or an interpretation. An interpretation $\mu(= \bar{\Sigma}$ for some $\Sigma$) satisfies a set $\Delta$ is denoted $\mu \models \Delta$. The validity $\Sigma \models \varphi$ is defined as: For any interpretation $\mu$, $\mu \models \Sigma$ implies $\mu \models \{ \varphi \}$.

Then what kind of proof systems will have the extended completeness theorem? Consider $\Sigma \models \varphi$. It means that for any $\mu$ such that $\Sigma \subseteq \mu$, $\mu \vdash \varphi$ (that is, $\varphi \in \mu$). In other words, $\Sigma \models \varphi$ means $\varphi \in \bigcap_{\Sigma \subseteq \mu} \mu$. On the other hand, $\Sigma \vdash \varphi$ means $\varphi \in \bar{\Sigma}$.

To have both soundness and completeness hold, we need $\bigcap_{\Sigma \subseteq \mu} \mu = \bar{\Sigma}$. To have this, the reflexivity condition $\Sigma \subseteq \bar{\Sigma}$ implies completeness. And the transitivity condition that “For any interpretation $\mu$ and any set $\Sigma$, if $\Sigma \subseteq \mu$, then $\bar{\Sigma} \subseteq \mu$” implies soundness.

\textsuperscript{3}And this folklore construction is probably well-known to most logicians. One just simply forces the validity to be the provability.
Theorem 9. If $\vdash$ satisfies the reflexivity condition (for any set $\Sigma$ and $\varphi$, $\varphi \in \Sigma$ implies $\Sigma \vdash \varphi$) and the transitivity condition (for any sets $\Delta, \Sigma$, and any sentence $\tau$, if $\Delta \vdash \varphi$ for every $\varphi \in \Sigma$ and $\Sigma \vdash \tau$, then $\Delta \vdash \tau$), then to the corresponding semantics (as constructed above), $\vdash$ is sound and complete.

Can we say that the compactness theorem a trivial result since for any given finitary, reflexive, transitive proof system one can simply apply Theorems 9,8 to prove the compactness theorem? The answer is NO. The problem of this $\vdash$-based semantics is that it may not fully reflect what we expect it supposed to be. For example, given a proof system $CPL$, it is not clear why the corresponding folklore semantics is the two-valued truth-functional semantics (though they can be proved equivalent) and we should satisfy with the compactness theorem with respect to this folklore semantics.

To end this paper I would like to explain that my proof (see [9] or [10]) that $IPL$ also satisfies the classical model existence theorem should be applicable to show that the Kripke semantics of $IPL$ has the compactness theorem. Unlike the case in Theorem 9 here, one can show that a classical model is also a intuitionistic Kripke model (by checking the definition of the Kripke semantics of $IPL$). Therefore the model existence theorem of $IPL$ holds and the compactness theorem is proved by Theorem 8.

Concluding Remark: It is a pity then I do not show any nice exact condition for semantics having compactness theorem. However, I hope that it may be satisfactory for some readers that I actually offer (sketches of)
proofs of compactness theorem for finite-valued propositional semantics and
$IPL$ (and some sufficient conditions in abstract forms).

4. Appendix

It is known that the classical (in)consistency is equivalent to the intuitionistic (in)consistency (see [14]). Except the proof theoretic approach in [14], here we give a quick proof of it by employing the well-known Glivenko’s theorem (if $\Sigma \vdash_{CPL} \varphi$, then $\Sigma \vdash_{IPL} \neg \neg \varphi$). If $\Sigma \vdash_{CPL} \bot$, then by Glivenko’s theorem we have $\Sigma \vdash_{IPL} \neg \neg \bot$. Since $\neg \neg \bot$ is $(\bot \rightarrow \bot) \rightarrow \bot$, $\bot \rightarrow \bot$ is provable in $IPL$, and modus ponens holds in $IPL$, we have $\Sigma \vdash_{IPL} \bot$. (The other direction is trivial since $IPL$ is a sublogic of $CPL$.)

Remark 10. This is actually a reply to two referees who failed to recognize the mathematical correctness of my result (in a revised version of my TPA2004 paper).

References


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